Example (Ex 1.2.2 in book)

Prove that there is no rational # r such that $2^r = 3$.

Thoughts - we'll suppose $r = \frac{a}{b}$, i.e. rational ($a \in \mathbb{Z}$, $b \in \mathbb{N}$)

$2^{a/b} = 3$, then hopefully get contradiction

$[2^{a/b}]^b = 3^b \Leftrightarrow 2^a = 3^b$

if $a<0$, $2^a<1$, $\times$

if $a=0$, $2^a=1$, $\times$

if $a>0$, by uniqueness of prime f. $\times$.

$\log(2^r) = \log 3$

$r \log(2) = \log(3)$

$r = \frac{\log(3)}{\log(2)}$ Not helpful

(Clean version)

Proof: Suppose $r \in \mathbb{Q}$, so that $r = \frac{a}{b}$ with $a \in \mathbb{Z}$, $b \in \mathbb{N}$.* We now assume $2^r = 2^{a/b} = 3$.

Then $2^a = (2^{a/b})^b = 3^b$.

If $a<0$, $2^a < 1 < 3^b$, a contradiction.

If $a=0$, $2^a = 1 < 3^b$, a contradiction.

If $a>0$, 2^a and 3^b are different prime factorizations of the same number. This is a contradiction, because prime factorization is unique up to the order of primes. We have shown that the equation $2^a = 3^b$ cannot be solved by $a \in \mathbb{Z}$ and $b \in \mathbb{N}$.

Thus, the assumption that $2^r = 3$ cannot hold for rational r. $\boxtimes$

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* Note that every rational number is a ratio of integers $\frac{x}{y}$, with $y \neq 0$. If $y < 0$, we may also write this number as $\frac{-x}{|y|}$, so the denominator can always be assumed to be positive.

Example (Ex 1.2.3) Prove or disprove each of the following.

(a) If $A_1, A_2, \ldots$ are each infinite sets and if $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$

Then $\bigcap_{n=1}^{\infty} A_n$ is an infinite set.

Thoughts $\bigcap_{n=1}^{\infty} A_n = \{x : x \in A_n \ \forall n \in \mathbb{N}\}$

"the set of all x such that x is in A_n for all n"

Let $A_1 = \mathbb{N} = \{1, 2, 3, \ldots\}$, $A_2 = 2\mathbb{N} = \{2, 4, 6, \ldots\}$

$A_3 = 4\mathbb{N} = \{4, 8, 12, \ldots\}$, $\ldots$ ie $A_j = 2^{j-1}\mathbb{N}$

$A_1 = \mathbb{N}$, $A_2 = \mathbb{N}+1 = \{2,3,4,\ldots\}$, $A_3 = \mathbb{N}+2$
$= \{3,4,5\ldots$

$A_j = \mathbb{N}+j-1.$

How do we show

$$\bigcap_{n=1}^{\infty} A_n = \emptyset \,?$$

Well, $A_1 \supseteq \bigcap_{n=1}^{\infty} A_n$, so $\bigcap_{n=1}^{\infty} A_n \subseteq \mathbb{N}$.

If $k \in \mathbb{N}$, then $k \notin A_{k+1}$, so $k \notin \bigcap_{n=1}^{\infty} A_n$.

<u>Proof:</u> The statement is false.

Define $A_j = \mathbb{N}+j-1$ for all $j \in \mathbb{N}$.

Since $A_{j+1} \cup \{j\} = A_j$, $A_{j+1} \subseteq A_j$ for all $j \in \mathbb{N}$, and each A_j is infinite.

If $k \in \mathbb{N}$, then $k \notin A_{k+1}$, so $k \notin \bigcap_{n=1}^{\infty} A_n$.

But $\bigcap_{n=1}^{\infty} A_n \subseteq A_1 = \mathbb{N}$. Thus $\bigcap_{n=1}^{\infty} A_n = \emptyset$. $\blacksquare$

Thus, $\{A_1, A_2, \ldots\}$ provides a counterexample.

(b) Suppose $A_1, A_2, \ldots$ are nonempty, finite sets, such that

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots.$$

Then $\bigcap_{n=1}^{\infty} A_n$ is nonempty and finite.

Thoughts Definitely $\bigcap_{n=1}^{\infty} A_n$ must be finite (all inside A_1)

- If $|A_j| = |A_{j+1}|$ then $A_j = A_{j+1}$

  (since $A_{j+1} \subseteq A_j$, if any element is left out $|A_{j+1}| < |A_j|$)

- How many times can $|A_{j+1}| < |A_j|$?

Ans: No more than $|A_1|$.

Thus, there are only a finite number of different sets in $\{A_j\}$.

$\Rightarrow \exists N \in \mathbb{N}$ s.t. $A_N = A_{N+1} = A_{N+2} = \ldots$

Then $A_N \subseteq \bigcap_{j=1}^{\infty} A_j$. and A_N is nonempty & finite.

Mathy way to say this:

"The sets A_j stabilize eventually."

Proof: (left as exercise (HW))

prove or Disprove

(c) Suppose A, B, C are sets.

Then $A \cap (B \cup C) = (A \cap B) \cup C$.

Thoughts: Really $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Example: $A = \{1\}$, $B = \{2\}$, $C = \{2,3\}$

$LHS = \{1\} \cap (\{2,3\}) = \emptyset$

$RHS = \emptyset \cup \{2,3\} = \{2,3\}$.

Proof: The statement is false.

Let $A = \{1\}$, $B = \{2\}$, $C = \{2,3\}$.

Then $A \cap (B \cup C) = \{1\} \cap \{\{2\} \cup \{2,3\}\}$

$= \{1\} \cap \{2,3\} = \emptyset$.

and $(A \cap B) \cup C = (\{1\} \cap \{2\}) \cup \{2,3\}$

$= \emptyset \cup \{2,3\} = \{2,3\}$

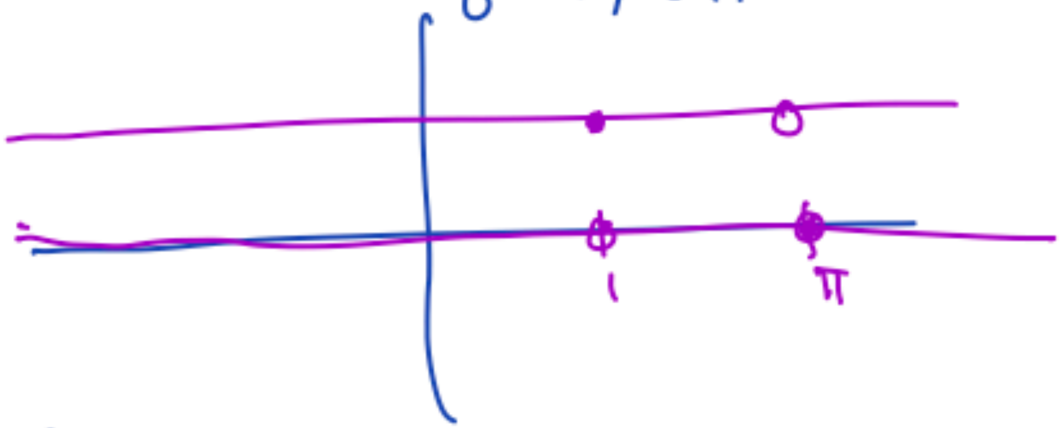
$\neq A \cap (B \cup C)$. $\blacksquare$

This is a counterexample.

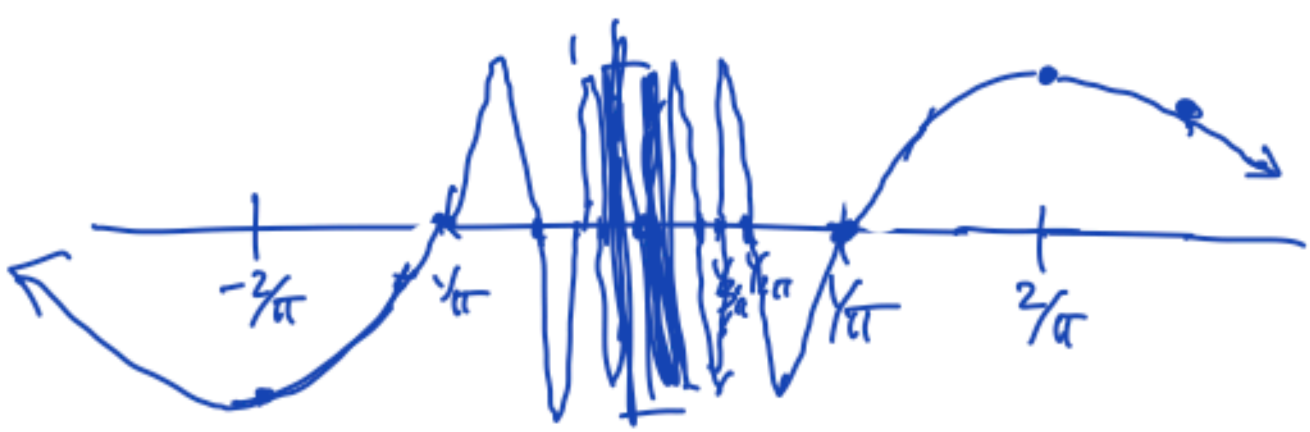
Interesting examples of functions

① Dirichlet Function $D: \mathbb{R} \to \mathbb{R}$

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$



② $$F(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



Example (Ex. 1.2.8) Give an example, or show that no such example is possible.

ⓐ function $f: \mathbb{N} \to \mathbb{N}$ that is 1-1 and not onto

Thoughts 1-1 would mean for all $x, y \in \mathbb{N}$ (range), if $f(x) = f(y)$ then $x = y$.

onto would mean $\forall y \in \mathbb{N}$ (range), $\exists x \in \mathbb{N}$ (domain) s.t. $f(x) = y$.

Ex: $f(x) = 2x$. ....